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A FUZZY GH-FINITE DIFFERENCE NEWTON METHOD FOR SOLVING FUZZY NONLINEAR OPTIMIZATION PROBLEMS

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ABSTRACT. This paper introduces a new numerical technique for solving nonlinear fuzzy programs. This novel approach is of the Newton type and is founded on the gH-finite difference of fuzzy nonlinear functions. It finds a non-dominated solution to a multivariate, nonlinear, fuzzy optimization problem by combining the gH-difference concept for fuzzy quantities, the finite difference notion, and Newton's algorithm. The process begins with the formulation of a finite difference gH of multivariate fuzzy nonlinear functions via gH-differentiability. This new method has the unique ability to efficiently handle optimization problems involving fuzzy nonlinear functions with highly complex derivatives. Unlike the fuzzy Newton methods presented in the literature, the method presented in this paper does not use the notion of derivative. Rather, it uses the fuzzy gH-finite difference technique to approximate the gradient and Hessian matrix of nonlinear fuzzy functions. Therefore, it applies to fuzzy nonlinear optimization problems for which the first and second partial gH-derivatives are difficult to obtain through classical calculation.

1. INTRODUCTION

Solving an optimization problem with linear or nonlinear constraints—or with no constraints at all—means finding a set of variables that minimizes or maximizes the objective function's value while respecting the constraints, if they exist. The existence of solutions to such a problem requires that the functions be differentiable. Several numerical methods have been developed to solve this type of problem [1–6], in addition to the exact methods . Among them, Newton's method is remembered. Newton's method [7–9], also called the Newton-Raphson method, is an iterative algorithm for numerically determining a precise approximation of the optimum of a function with real variables. This method was initially proposed for solving polynomial equations, and its relevance in the search for the optimum has aroused interest among several researchers. This is illustrated by [10], who, in his work, uses the notion of derivatives to solve nonlinear equations. When the starting point is far from the local optimum, this local method presents convergence difficulties. Given these conditions, it cannot be determined whether the Hessian matrix at this point is positive definite or if the Newton direction is a descent direction. Many modifications of this method have been proposed to overcome all these shortcomings [11–16]. It is a method that relies heavily on the concept of derivatives. However, calculating the derivative of certain functions is expensive. Thus, J. K. Cayford [17] raises this limit by using the finite difference method to approximate the derivative of the

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function in order to solve the two-center electronic Schrödinger equation. Newton's method was quickly adapted for use with functions of multiple variables [18]. Its effectiveness in nonlinear deterministic multi-objective optimization has been demonstrated through the work of Fliege et al. [19], Qinwu Xu [20], and Žiga Povalej [21]. This method contributed to advancements in optimization in the deterministic case. Its performance was also tested in the fuzzy number space.

Newton's method first appeared in the field of fuzzy optimization in 2004 in the work of Abbasbandy and Asady [22], who used it to solve a nonlinear fuzzy problem. In 2013, Pirzada and Pathak [23] combined the H-difference method and Newton's method to find a non-dominated solution to an unconstrained, multivariate, nonlinear function optimization problem. In 2015, Chalco-Cano, Silva, and Rufián-Lizana [24] identified a limitation in the work of [23] and proposed a new version of Newton's method to solve unconstrained, multivariate, fuzzy, nonlinear optimization problems using the gH-difference. In 2016, Ghosh [25] extended it to the optimization of nonlinear interval-valued functions. In 2020, Umar et al. [26] proposed the fuzzy Quasi-Newton variant. Until now, variants of Newton's method proposed for fuzzy nonlinear optimization have relied on the gradient and Hessian matrix of the fuzzy function. Calculating these matrices is often difficult, especially for functions derived from experiments or observations. In this sense, this work proposes a new approach based on the fuzzy Newton method. The method uses Samaneh Zabihi's [27] proposal, which extends the Taylor formula to fuzzy, nonlinear functions of one variable, to define gH-finite differences of fuzzy functions of several variables. It also proposes an extension to Newton's method using these differences. The goal is to replace the gradient and Hessian matrix of the fuzzy Newton method with their fuzzy gH-finite difference approximations.

This method is an improvement on those proposed in the literature. If the function is not H-differentiable or calculating the gradient is costly, this method circumvents these issues and provides a fuzzy approximation of the gradient. Furthermore, rather than directly using the expression of α-cuts, whose number increases with the number of variables, the proposed method uses the supports of these numbers. This avoids the disadvantages resulting from multiplying the number of α-cuts by sign-free variables.

To present the results of this work more effectively, we will use the following structure:
Section 2: Presentation of the essential notions necessary for understanding the subsequent discussions.
Section 3: Exposition of the main results, namely fuzzy finite differences, the extension of Newton's method to fuzzy finite differences, the study of its convergence, examples of resolution, and a possible discussion.
Section 4 will present the conclusion that summarizes the contributions of this article.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers. $[\tilde{u}]_\alpha$ denotes the set of $\alpha-$ cuts of a fuzzy triangular number $\tilde{u} = (a, b, c)$ with a, b and c real numbers such that $a \leq b \leq c$, by

$$[\tilde{u}]_\alpha = [u_\alpha^-, u_\alpha^+] = [\alpha(b-a)+a, \alpha(b-c)+c], \forall \alpha \in [0,1].$$

Definition 2.1. [28]
Let K be a subset of $\mathbb{R}^n$. Any function $\tilde{f} : K \to \mathbb{R}_{\mathcal{F}}$ is called a fuzzy function with real variables.

Definition 2.2. [24]
Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy numbers and
$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$. A distance between $\tilde{u}$ and $\tilde{v}$ is defined as follows:

$$\begin{aligned} D(\tilde{u}, \tilde{v}) &= \sup_{\alpha \in [0,1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\} \\ &= \sup_{\alpha \in [0,1]} \{d_H([\tilde{u}]_\alpha, [\tilde{v}]_\alpha)\}, \end{aligned}$$

where $d_H = \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}$ is the Hausdorff Pompeiu distance [29] between the intervals of the α-cuts.

Proposition 2.3. *[30]*
A fuzzy norm $\|.\|_{\mathcal{F}}$ is an application of $\mathbb{R}_{\mathcal{F}} \to \mathbb{R}$ that has the following properties.

(i) $\|\tilde{u}\|_{\mathcal{F}} = 0$ *if and only if* $\tilde{u} = \tilde{0}$.
(ii) $\|\lambda \odot \tilde{u}\|_{\mathcal{F}} = |\lambda|.\|\tilde{u}\|_{\mathcal{F}}, \quad \forall \lambda \in \mathbb{R}$ *and* $\tilde{u} \in \mathbb{R}_{\mathcal{F}}$.
(iii) $\|\tilde{u} \oplus \tilde{v}\|_{\mathcal{F}} \leq \|\tilde{u}\|_{\mathcal{F}} + \|\tilde{v}\|_{\mathcal{F}}\ , \quad \forall \tilde{u}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$.
(iv) $\|\tilde{u}\|_{\mathcal{F}} - \|\tilde{v}\|_{\mathcal{F}} \leq D(\tilde{u}, \tilde{v})\ , \quad \forall \tilde{u}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$.
(v) $D(\tilde{u}, \tilde{v}) = \|\tilde{u} \ominus_{gH} \tilde{v}\|_{\mathcal{F}}$.

In Proposition 2.3, $\ominus_{gH}$ denotes the gH-difference defined as follows:

Definition 2.4. [31]
Let two triangular fuzzy numbers
$\tilde{u} = (u_1, u_2, u_3)$ and $\tilde{v} = (v_1, v_2, v_3)$ be given. The generalized Hukuhara difference (gH-difference) of the numbers $\tilde{u}$ and $\tilde{v}$ is the fuzzy number
$\tilde{w} = (w_1, w_2, w_3)$ (if it exists) such that:

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} = \begin{cases} (w_1, w_2, w_3), \\ or \\ (w_3, w_2, w_1), \end{cases}$$
$$= \begin{cases} (u_1 - v_1, u_2 - v_2, u_3 - v_3), \\ or \\ (u_3 - v_3, u_2 - v_2, u_1 - v_1). \end{cases}$$

In terms of $\alpha-$cut, we have:

$$[\tilde{u} \ominus_{gH} \tilde{v}]_\alpha = [\min\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}, \max\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}], \forall \alpha \in [0, 1].$$

Definition 2.5. [32]
Let $\Upsilon : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}$ be a ranking function and $\tilde{u} = (u_1, u_2, u_3)$ be a triangular fuzzy number, then

$$\Upsilon(\tilde{u}) = \frac{u_1 + 2u_2 + u_3}{4}.$$

Definition 2.6. Let $\mathcal{A}$ be a matrix with triangular fuzzy components defined as follows:

$$\tilde{\mathcal{A}} := \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \cdots & \tilde{a}_{mn} \end{pmatrix}.$$

The infinite fuzzy matrix norm is defined as follows: $\tilde{\mathcal{A}} = \max_{1 \leq i \leq m} \left\{ \bigoplus_{j=1}^{n} \tilde{a}_{ij} \right\}$ via the ranking function, where $\bigoplus_{j=1}^{n} \tilde{a}_{ij}$ denotes the sum of the numbers $\tilde{a}_{ij}$.

3. Main results

In this section, we first present a gH-finite difference method for multivariate fuzzy nonlinear functions. Then, we introduce a Newton gH-finite difference algorithm for solving unconstrained fuzzy nonlinear optimization problems.

3.1. **gH-fuzzy finite difference.** Let $K \subseteq \mathbb{R}^n$ be $\mathcal{C}^2_{gH}(K, \mathbb{R}_{\mathcal{F}})$, the set of twice gH-continuously differentiable functions from K to $\mathbb{R}_{\mathcal{F}}$. Let $\tilde{f} \in \mathcal{C}^2_{gH}(K, \mathbb{R}_{\mathcal{F}})$ be a twice gH-continuously differentiable nonlinear fuzzy function, $h \in \mathbb{R}^*_+$ such that $x \pm he_i \in K$ with e_i, $i = \overline{1, n}$, vectors of the canonical basis.

Proposition 3.1.
Let us define the gH-Taylor expansion of multivariate fuzzy functions as follows:

$$\tilde{f}(x + he_i) = \tilde{f}(x) \oplus (he_i)^T \odot \tilde{\nabla}_{gH}\tilde{f}(x) \oplus \frac{1}{2}(he_i)^T \odot \tilde{\nabla}^2_{gH}\tilde{f}(x) \odot (he_j) \oplus O(h^2), \tag{3.1}$$

$$\tilde{f}(x - he_i) = \tilde{f}(x) \ominus (he_i)^T \odot \tilde{\nabla}_{gH}\tilde{f}(x) \oplus \frac{1}{2}(he_i)^T \odot \tilde{\nabla}^2_{gH}\tilde{f}(x) \odot (he_j) \ominus O(h^2), \tag{3.2}$$

where

$$\tilde{\nabla}_{gH}\tilde{f}(x) = \left(\frac{\partial_{gH}\tilde{f}(x)}{\partial x_1}, \frac{\partial_{gH}\tilde{f}(x)}{\partial x_2}, \cdots, \frac{\partial_{gH}\tilde{f}(x)}{\partial x_n} \right)$$

and

$$\tilde{\nabla}^2_{gH}\tilde{f}(x) = \left(\frac{\partial^2_{gH}\tilde{f}}{\partial x_i \partial x_j} \right)_{1 \le i,j \le n}.$$

Definition 3.2.
From (3.1), we define the right fuzzy gH-finite difference as follows:

$$\left[\frac{\partial_{gH}\tilde{f}(x)}{\partial x_i} \right]^L \approx \frac{\tilde{f}(x + he_i) \ominus_{gH} \tilde{f}(x)}{h}. \tag{3.3}$$

From (3.2), we also define the left gH-finite difference as follows:

$$\left[\frac{\partial_{gH}\tilde{f}(x)}{\partial x_i} \right]^R \approx \frac{\tilde{f}(x) \ominus_{gH} \tilde{f}(x - he_i)}{h}. \tag{3.4}$$

By doing (3.1) - (3.2), we define the centered gH-finite difference as follows:

$$\left[\frac{\partial_{gH}\tilde{f}(x)}{\partial x_i} \right]^C \approx \frac{\tilde{f}(x + he_i) \ominus_{gH} \tilde{f}(x - he_i)}{2h}. \tag{3.5}$$

Note that these fuzzy gH-finite differences will be used to approximate the gradient of a gH-differentiable fuzzy function.

Theorem 3.3.
Let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}_{\mathcal{F}}$ be a twice gH-continuously differentiable nonlinear fuzzy function. Assume that the n-th gH-partial derivatives of $\tilde{f}$ are bounded by $\tilde{\gamma}$. So, we have:

$$\|\tilde{f}(x + he_i) \ominus_{gH} \tilde{f}(x) \ominus_{gH} (he_i)^T \tilde{\nabla}_{gH}\tilde{f}(x) \ominus_{gH} \frac{1}{2}(he_i)^T \tilde{\nabla}^2_{gH}\tilde{f}(x) he_i\|_{\mathcal{F}} \le \frac{h^3}{6}\|\tilde{\gamma}\|_{\mathcal{F}}. \tag{3.6}$$

Proof. Let us adopt the idea of the Taylor expansion of a fuzzy function as follows:

$$\tilde{f}(x + he_i) \cong$$

$$\tilde{f}(x) \oplus (he_i)^T \odot \tilde{\nabla}_{gH}\tilde{f}(x) \oplus \frac{1}{2}(he_i)^T \odot \tilde{\nabla}^2_{gH}\tilde{f}(x) \odot (he_i) \oplus \frac{1}{6}\sum_{i,j,k} \frac{\partial^3_{gH}\tilde{f}}{\partial x_i \partial x_j \partial x_k}(\xi)\, he_i\, he_j\, he_k.$$

So

$$\tilde{f}(x+he_i)\ominus_{gH}\tilde{f}(x)\ominus_{gH}(he_i)^T\odot\tilde{\nabla}_{gH}\tilde{f}(x)\ominus_{gH}\frac{1}{2}(he_i)^T\odot\tilde{\nabla}^2_{gH}\tilde{f}(x)\odot(he_i)\cong$$
$$\frac{1}{6}\sum_{i,j,k}\frac{\partial^3_{gH}\tilde{f}}{\partial x_i\partial x_j\partial x_k}(\xi)\,he_i\,he_j\,he_k,$$

With $\xi\in[x,x+h]$ and $\frac{1}{6}\sum_{i,j,k}\frac{\partial^3_{gH}\tilde{f}}{\partial x_i\partial x_j\partial x_k}(\xi)\,he_i\,he_j\,he_k$, represents the error of the third-order gH-Taylor expansion.

Subsequently, we have:

$$|\tilde{f}(x+he_i)\ominus_{gH}\tilde{f}(x)\ominus_{gH}\tilde{\nabla}\tilde{f}(x)^The_i\ominus_{gH}\frac{1}{2}(he_i)^T\tilde{\nabla}^2\tilde{f}(x)he_i\|_{\mathcal{F}}\leq\frac{h^3}{6}\|\sum_{i,j,k}\frac{\partial^3_{gH}\tilde{f}}{\partial x_i\partial x_j\partial x_k}(\xi)\|_{\mathcal{F}}. \tag{3.7}$$

Hence

$$\|\tilde{f}(x+he_i)\ominus_{gH}\tilde{f}(x)\ominus_{gH}(he_i)^T\tilde{\nabla}_{gH}\tilde{f}(x)\ominus_{gH}\frac{1}{2}(he_i)^T\tilde{\nabla}^2_{gH}\tilde{f}(x)he_i\|_{\mathcal{F}}\leq\frac{h^3}{6}\|\tilde{\gamma}\|_{\mathcal{F}}.$$

□

Theorem 3.4. *Let $\tilde{f}:K\subseteq\mathbb{R}^n\to\mathbb{R}_{\mathcal{F}}$ be a gH-differentiable fuzzy function. Suppose that for all $h\in\mathbb{R}^*_+$, we have: $x\pm he_i\in K$, $i=1,...,n$. Also, let us define the fuzzy vector $\tilde{G}$ with its fuzzy components $\tilde{g}_i$ defined as follows:*

$$\tilde{g}_i(x)=\frac{\tilde{f}(x+he_i)\ominus_{gH}\tilde{f}(x-he_i)}{2h}; \tag{3.8}$$

So, we have:

$$\|\tilde{g}_i(x)\ominus_{gH}[\tilde{\nabla}_{gH}\tilde{f}(x)]_i\|_{\mathcal{F}}\leq\frac{h^2}{6}\|\tilde{\gamma}\|_{\mathcal{F}}, \tag{3.9}$$

where $[\tilde{\nabla}_{gH}\tilde{f}(x)]_i$ is the i-th component of the gH-gradient of the fuzzy function $\tilde{f}$.

Further, we have:

$$\|\tilde{G}(x)\ominus_{gH}\tilde{\nabla}_{gH}\tilde{f}(x)\|_{\mathcal{F}}\leq\frac{h^2}{6}\|\tilde{\gamma}\|_{\mathcal{F}}. \tag{3.10}$$

Proof. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be defined as follows:

$$\tilde{\alpha}(x)=\tilde{f}(x+he_i)\ominus_{gH}\tilde{f}(x)\ominus_{gH}(he_i)^T\odot\tilde{\nabla}_{gH}\tilde{f}(x)\quad\ominus_{gH}\frac{1}{2}(he_i)^T\odot\tilde{\nabla}^2_{gH}\tilde{f}(x)\odot(he_i);$$
$$\tilde{\beta}(x)=\tilde{f}(x-he_i)\ominus_{gH}\tilde{f}(x)\oplus(he_i)^T\odot\tilde{\nabla}_{gH}\tilde{f}(x)\ominus_{gH}\frac{1}{2}(he_i)^T\odot\tilde{\nabla}^2_{gH}\tilde{f}(x)\odot(he_i).$$

From these last two equalities, we have:

$$\tilde{\alpha}(x)\ominus_{gH}\tilde{\beta}(x)=2h(\tilde{g}_i(x)\ominus_{gH}[\tilde{\nabla}_{gH}\tilde{f}(x)]_i).$$

Using (iii) of Proposition 2.3, we have:

$$\|\tilde{\alpha}\ominus_{gH}\tilde{\beta}\|_{\mathcal{F}}\leq\|\tilde{\alpha}\|_{\mathcal{F}}+\|\tilde{\beta}\|_{\mathcal{F}}.$$

From (3.6) we also have:

$$\|\tilde{\alpha}\|_{\mathcal{F}}\leq\frac{h^3}{6}\|\tilde{\gamma}\|_{\mathcal{F}}\text{ and }\|\tilde{\beta}\|_{\mathcal{F}}\leq\frac{h^3}{6}\|\tilde{\gamma}\|_{\mathcal{F}}.$$

Which leads to that:

$$\|\tilde{\alpha}\|_{\mathcal{F}}+\|\tilde{\beta}\|_{\mathcal{F}}\leq\frac{h^3}{3}\|\tilde{\gamma}\|_{\mathcal{F}}.$$

Subsequently,

$$\|2h(\tilde{g}_i \ominus_{gH} [\tilde{\nabla}_{gH}\tilde{f}(x)]_i)\|_{\mathcal{F}} \leq \frac{h^3}{3}\|\tilde{\gamma}\|_{\mathcal{F}}.$$

Hence,

$$\|\tilde{g}_i \ominus_{gH} [\tilde{\nabla}_{gH}\tilde{f}(x)]_i\|_{\mathcal{F}} \leq \frac{h^2}{6}\|\tilde{\gamma}\|_{\mathcal{F}}. \tag{3.11}$$

Using the infinite matrix norm on $\tilde{G}$ and on $\tilde{\nabla}_{gH}\tilde{f}(x)$, we obtain (3.10). □

Theorem 3.5. *Let us assume a fuzzy function $\tilde{f}$ satisfying the conditions of Theorem 3.4 with x, $x+he_i+he_j$, $x+he_i$, $x+he_j \in K$, $1 \leq i,j \leq n$ and $\| e_i \| = 1$. Let us also define the square matrix with fuzzy coefficients of order n, $\tilde{A}$, whose components are presented as follows:*

$$\tilde{a}_{ij}(x) = \frac{\tilde{f}(x+he_i+he_j) \oplus \tilde{f}(x) \ominus_{gH} \tilde{f}(x+he_j) \ominus_{gH} \tilde{f}(x+he_i)}{h^2}.$$

So

$$\|\tilde{a}_{ij}(x) \ominus_{gH} [\tilde{\nabla}^2_{gH}\tilde{f}(x)]_{ij}\|_{\mathcal{F}} \leq \frac{5}{3}h\|\tilde{\gamma}\|_{\mathcal{F}}. \tag{3.12}$$

More generally, we have:

$$\| \tilde{A}(x) \ominus_{gH} \tilde{\nabla}^2_{gH}\tilde{f}(x)\|_{\mathcal{F}} \leq \frac{5}{3}h\|\tilde{\gamma}\|_{\mathcal{F}}. \tag{3.13}$$

Proof. Consider the following equalities:

$\tilde{\alpha}(x) = \tilde{f}(x+he_i+he_j) \ominus_{gH} \tilde{f}(x) \ominus_{gH} \tilde{\nabla}_{gH}\tilde{f}(x)^T(he_i+he_j) \ominus_{gH} \frac{1}{2}(he_i+he_j)^T\tilde{\nabla}^2_{gH}\tilde{f}(x)(he_i+he_j)$,

$\tilde{\beta}(x) = \tilde{f}(x+he_i) \ominus_{gH} \tilde{f}(x) \ominus_{gH} \tilde{\nabla}_{gH}\tilde{f}(x)^T he_i \ominus_{gH} \frac{1}{2}(he_i)^T\tilde{\nabla}^2_{gH}\tilde{f}(x)he_i$,

$\tilde{\eta}(x) = \tilde{f}(x+he_j) \ominus_{gH} \tilde{f}(x) \ominus_{gH} \tilde{\nabla}_{gH}\tilde{f}(x)^T he_j \ominus_{gH} \frac{1}{2}(he_j)^T\tilde{\nabla}^2_{gH}\tilde{f}(x)he_j$.

By doing $\tilde{\alpha}(x) \ominus_{gH} \tilde{\beta}(x) \ominus_{gH} \tilde{\eta}(x)$, we get:

$$\tilde{\alpha}(x) \ominus_{gH} \tilde{\beta}(x) \ominus_{gH} \tilde{\eta}(x) = h^2(\tilde{a}_{ij} \ominus_{gH} [\tilde{\nabla}^2_{gH}\tilde{f}(x)]_{ij}). \tag{3.14}$$

From (iii) of Proposition 2.3, we have:

$$\|\tilde{\alpha}(x) \ominus_{gH} \tilde{\beta}(x) \ominus_{gH} \tilde{\eta}(x)\|_{\mathcal{F}} \leq \|\tilde{\alpha}(x)\|_{\mathcal{F}} + \|\tilde{\beta}(x)\|_{\mathcal{F}} + \|\tilde{\eta}(x)\|_{\mathcal{F}}.$$

From (3.6) and (3.15), we have:

$$\begin{aligned}\|\tilde{\alpha}(x)\|_{\mathcal{F}} + \|\tilde{\beta}(x)\|_{\mathcal{F}} + \|\tilde{\eta}(x)\|_{\mathcal{F}} &\leq \frac{\gamma}{6}\|he_i+he_j\|^3 + \frac{\gamma}{6}\|he_i\|^3 + \frac{\gamma}{6}\|he_j\|^3 \\ &\leq \frac{5}{3}h^3\|\tilde{\gamma}\|_{\mathcal{F}}.\end{aligned}$$

Subsequently, we have:

$$\|h^2(\tilde{a}_{ij} \ominus_{gH} [\tilde{\nabla}^2_{gH}f(x)]_{ij})\|_{\mathcal{F}} \leq \frac{5}{3}h^3\|\tilde{\gamma}\|_{\mathcal{F}}. \tag{3.15}$$

Hence

$$\|(\tilde{a}_{ij} \ominus_{gH} [\tilde{\nabla}^2_{gH}f(x)]_{ij})\|_{\mathcal{F}} \leq \frac{5}{3}h\|\tilde{\gamma}\|_{\mathcal{F}}. \tag{3.16}$$

Using the infinite matrix norm on $\tilde{A}$ and on $\tilde{\nabla}^2_{gH}f(x)$, we obtain (3.13). □

Thus, the fuzzy gradient and the fuzzy Hessian matrix can be approximated by $\tilde{G}$ and $\tilde{A}$, respectively.

3.2. **Newton's method with fuzzy gH-finite difference.** Consider the following unconstrained optimization problem:

$$\min_{x\in\mathbb{R}^n} \tilde{f}(x) \tag{3.17}$$

where $\tilde{f}$ is a twice gH-continuously differentiable nonlinear fuzzy function with real variables.
Let $x^s \in \mathbb{R}^n$ be such that $x^s + he_j \in \mathbb{R}^n$, $\forall\, h \in \mathbb{R}_+^*$. Suppose that the generalized Hukuhara difference between $\tilde{f}(x^s + he_j)$ and $\tilde{f}(x^s)$ exists.

To solve such a problem, the proposed method uses three steps, namely:

Step I: Approximation of the gH-fuzzy gradient and the gH-fuzzy Hessian matrix from the gH-finite differences:

$$(\tilde{g})_j(x^s) = \frac{\tilde{f}(x^s + he_j) \ominus_{gH} \tilde{f}(x^s)}{h},\ j = 1, 2, \cdots, n; \tag{3.18}$$

$$(\tilde{a})_{ij}(x^s) = \frac{\tilde{f}(x^s + he_j + he_i) \ominus_{gH} \tilde{f}(x^s + he_j) \ominus_{gH} \tilde{f}(x^s + he_i) \oplus \tilde{f}(x^s)}{h^2}. \tag{3.19}$$

Step II: Defuzzification

From (3.18) and (3.19), we give the α-cut of each fuzzy coefficients of the problem in the form $[u_\alpha^-, u_\alpha^+]$.Then, each obtained α-cut is transformed into $\int_0^1 (u_\alpha^+ - u_\alpha^-)\, d\alpha$. We thus defuzzify $(\tilde{g})_j(x)$ and $(\tilde{a})_{ij}(x)$ to obtain the components $g_j(x)$ of the gradient $G(x)$ and $a_{ij}(x)$ of the Hessian matrix $A(x)$.

Step III: Resolution by Newton's method

Using the deterministic quantities G(x) and A(x), the iteration of Newton's method at the fuzzy gH-finite difference is formulated as follows:

$$x^{s+1} = x^s - [A(x^s)]^{-1} \times G(x^s), \tag{3.20}$$

where s represents the iteration number.

The algorithm of this method is as follows:

Algorithm 1 Newton's method with fuzzy gH-finite difference

```
1: Initialization: Enter the fuzzy function, choose x^0 ∈ R^n and ϵ > 0;
2: s ← 0;
3: Determine G(x^s) and A(x^s);
4: Repeat
5:     x^{s+1} = x^s − [A(x^s)]^{-1} × G(x^s);
6:     s ← s + 1;
7: Until
8:     ‖x^{s+1} − x^s‖ ≤ ϵ;
9: Return x̄^s.
```

3.3. **Study of Convergence.** In this section, we study the convergence of our method, which is an important step in ensuring the accuracy of our results.

Theorem 3.6. *Let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy nonlinear function with real variables twice gH-continuously differentiable and $x^* \in \mathbb{R}^n$ such that:*

(a) $G(x^*) = 0$,

(b) $A(x^*)$ *is regular,*

where $G(x^*) = \begin{pmatrix} g_1(x^*) \\ g_2(x^*) \end{pmatrix}$*, with* $\tilde{g}_i = \dfrac{\tilde{f}(x^* + he_i) \ominus_{gH} \tilde{f}(x^*)}{h}$ *and,* $A(x^*) = (a_{ij})_{1 \leq i,j \leq n}$ *with*

$$\tilde{a}_{ij} = \frac{\tilde{f}(x^s + he_j + he_i) \ominus_{gH} \tilde{f}(x^s + he_j) \ominus_{gH} \tilde{f}(x^s + he_i) \oplus \tilde{f}(x^s)}{h^2}.$$

Then, for x^0 *sufficiently close to* x^**, Newton's fuzzy gH-finite difference method is well defined for all* s *and converges to* x^* *with an order of convergence of at least 2.*

Proof. Using the Taylor expansion on $G(x)$ in the neighborhood of x^0, we have:

$$G(x) = G(x^0) + A(x^0).(x - x^0) + o(\|x - x^0\|^2).$$

Consequently, given a constant c_1, there exists $\zeta > 0$ such that for all $x^0 \in B(x^*, \zeta) = \{x, \|x - x^* \| \leq \zeta\}$, we have:

$$\|G(x) - G(x^0) - A(x^0).(x - x^0)\| \leq c_1 \|x - x^0\|^2. \tag{3.21}$$

Since $A(x^*)$ is regular, then $[A(x^*)]^{-1}$ is continuous at x^*; therefore, there exists a constant c_2 such that

$$\|A(x^*)\| < c_2.$$

And, since $G(x^*) = 0$, then for all $x \in B(x^*, \zeta)$, (3.21) becomes:

$$\|A(x^0).(x^0 - x) - G(x^0)\| \leq c_1 \|x - x^0\|^2. \tag{3.22}$$

Also, from (3.20), if $s = 0$, we have:

$$x^1 = x^0 - [A(x^0)]^{-1} \times G(x^0). \tag{3.23}$$

What gives

$$\begin{aligned} \|x^1 - x^*\| &= \|x^0 - x^* - [A(x^0)]^{-1} \times G(x^0)\| \\ &\leq \|[A(x^0)]^{-1}\|.\|[A(x^0)].(x^0 - x^*) - G(x^0)\| \\ &\leq c_1 c_2 \|x^0 - x^*\|^2, \end{aligned}$$

$c_1 c_2 \in]0, 1[$.

By choosing x^0 sufficiently close to x^* so that $\|x^0 - x^*\| < \dfrac{\beta}{c_1 c_2}$ with $\beta \in]0, 1[$, we have:

$$\|x^1 - x^*\| \leq \beta \|x^0 - x^*\|. \tag{3.24}$$

Iterating this process k times, we obtain:

$$\|x^{k+1} - x^*\| \leq c_1 c_2 \|x^k - x^*\|^2. \tag{3.25}$$

Afterwards

$$\|x^{k+1} - x^*\| \leq \beta \|x^k - x^*\|. \tag{3.26}$$

Which implies that $x^k \to x^*$ and we obtain a quadratic convergence from (3.25). □

3.4. **Didactic examples.** This section is devoted to examples taken from [23].

Example 3.7. Consider the following fuzzy problem:

$$\min \tilde{f}(x_1, x_2) = \tilde{1} \odot x_1^3 \ \oplus \ \tilde{2} \odot x_2^3 \ \oplus \ \tilde{1} \odot x_1 x_2, \tag{3.27}$$

where $x_1, x_2 \in \mathbb{R}$, $\tilde{1} = (-1, 1, 3)$ *and* $\tilde{2} = (1, 2, 3)$ are triangular fuzzy numbers.
Let us calculate $G(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2))^T$.
Let us start with the first component $g_1(x_1, x_2) = \dfrac{\tilde{f}(x_1 + h, x_2) \ominus_{gH} \tilde{f}(x_1, x_2)}{h}$, we have:

$$\tilde{f}(x_1 + h, x_2) = \tilde{1} \odot x_1^3 \oplus \tilde{1} \odot 3x_1^2 h \oplus \tilde{1} \odot 3x_1 h^2 \oplus \tilde{1} \odot h^3 \oplus \tilde{2} \odot x_2^3 \oplus \tilde{1} \odot x_1 x_2 \oplus \tilde{1} \odot x_2 h, \tag{3.28}$$

$$\tilde{f}(x_1 + h, x_2) \ominus_{gH} \tilde{f}(x_1, x_2) = \tilde{1} \odot 3x_1^2 h \oplus \tilde{1} \odot 3x_1 h^2 \oplus \tilde{1} \odot h^3 \oplus \tilde{1} \odot x_2 h. \tag{3.29}$$

From (3.28) and (3.29), we deduce

$$\tilde{g}_{1,h}(x_1, x_2) = \tilde{1} \odot 3x_1^2 \oplus \tilde{1} \odot 3x_1 h \oplus \tilde{1} \odot h^2 \oplus \tilde{1} \odot x_2.$$

Therefore, for h small enough, we have:
$\tilde{g}_1(x_1, x_2) = \tilde{1} \odot 3x_1^2 \oplus \tilde{1} \odot x_2$
So

$$g_{1,\alpha}(x_1, x_2) = [2\alpha - 1, -2\alpha + 3].3x_1^2 + [2\alpha - 1, -2\alpha + 3].x_2.$$

Therefore,

$$g_{1,\alpha}(x_1, x_2) = 3 \int_0^1 (-4\alpha + 4) x_1^2 \ d\alpha + \int_0^1 (-4\alpha + 4) x_2 \ d\alpha$$
$$g_1(x_1, x_2) = 6x_1^2 + 2x_2.$$

For the second component,
$\tilde{g}_2(x_1, x_2) = \dfrac{\tilde{f}(x_1, x_2 + h) \ominus_{gH} \tilde{f}(x_1, x_2)}{h}$, we have:

$$\tilde{f}(x_1, x_2 + h) = \tilde{1} \odot x_1^3 \oplus \tilde{2} \odot x_2^3 \oplus \tilde{2} \odot 3x_2^2 h \oplus \tilde{2} \odot 3x_2 h^2 \oplus \tilde{2} \odot h^3 \oplus \tilde{1} \odot x_1 x_2 \oplus \tilde{1} \odot x_1 h, \tag{3.30}$$

$$\tilde{f}(x_1, x_2 + h) \ominus_{gH} \tilde{f}(x_1, x_2) = \tilde{2} \odot 3x_2^2 h \oplus \tilde{2} \odot 3x_2 h^2 \oplus \tilde{2} \odot h^3 \oplus \tilde{1} \odot x_1 h. \tag{3.31}$$

From (3.30) and (3.31), we deduce:
$\tilde{g}_{2,h}(x_1, x_2) = \tilde{2} \odot 3x_2^2 \ \oplus \tilde{2} \odot 3x_2 h \ \oplus \tilde{2} \odot h^2 \ \oplus \tilde{1} \odot x_1$.
Therefore, for h small enough, we have:
$\tilde{g}_2(x_1, x_2) = \tilde{2} \odot 3x_2^2 \oplus \tilde{1} \odot x_1$.
We deduce that:

$$g_{2,\alpha}(x_1, x_2) = [\alpha + 1, -\alpha + 3].3x_2^2 + [2\alpha - 1, -2\alpha + 3].x_1,$$

Therefore,

$$g_{2,\alpha}(x_1, x_2) = 3 \int_0^1 (-2\alpha + 2).x_2^2 \ d\alpha + \int_0^1 (-4\alpha + 4).x_1 \ d\alpha$$
$$g_2(x_1, x_2) = 3x_2^2 + 2x_1.$$

Hence, $\quad G(x_1, x_2) = \begin{pmatrix} 6x_1^2 + 2x_2 \\ \\ 3x_2^2 + 2x_1 \end{pmatrix}$.

Let us calculate $A(x_1, x_2)$.
For the first component $a_{11}(x_1, x_2)$, we have:

$$\tilde{f}(x_1 + 2h, x_2) = \tilde{1} \odot x_1^3 \oplus \tilde{1} \odot 6x_1^2h \oplus \tilde{1} \odot 12x_1h^2 \oplus \tilde{1} \odot 8h^3 \oplus \tilde{2} \odot x_2^3 \oplus \tilde{1} \odot x_1x_2 \oplus \tilde{1} \odot 2x_2h, \quad (3.32)$$

$$\tilde{f}(x_1+2h, x_2) \oplus \tilde{f}(x_1, x_2) = \tilde{1} \odot 2x_1^3 \oplus \tilde{1} \odot 6x_1^2h \oplus \tilde{1} \odot 12x_1h^2 \oplus \tilde{1} \odot 8h^3 \oplus \tilde{2} \odot 2x_2^3 \oplus \tilde{1} \odot 2x_1x_2 \oplus \tilde{1} \odot 2x_2h, \quad (3.33)$$

$$2 \odot \tilde{f}(x_1 + h, x_2) = \tilde{1} \odot 2x_1^3 \oplus \tilde{1} \odot 6x_1^2h \oplus \tilde{1} \odot 6x_1h^2 \oplus \tilde{1} \odot 2h^3 \oplus \tilde{2} \odot 2x_2^3 \oplus \tilde{1} \odot 2x_1x_2 \oplus \tilde{1} \odot 2x_2h. \quad (3.34)$$

From (3.32), (3.33) and (3.34), we deduce:

$$\tilde{a}_{11,h}(x_1, x_2) = \tilde{1} \odot 6.x_1 \oplus \tilde{1} \odot 6.h.$$

For h small enough, we have:

$$\tilde{a}_{11}(x_1, x_2) = \tilde{1} \odot 6.x_1.$$

So

$$a_{11,\alpha}(x_1, x_2) = [2\alpha - 1, -2\alpha + 3].6x_1.$$

Subsequently, we obtain:

$$a_{11,\alpha}(x_1, x_2) = 6 \int_0^1 (-4\alpha + 4).x_1 d\alpha.$$

Hence,

$$a_{11}(x_1, x_2) = 12x_1.$$

Let us calculate the second and third components $a_{12}(x_1, x_2)$, and $a_{21}(x_1, x_2)$ of A.

$$\tilde{f}(x_1 + h, x_2 + h) = \tilde{1} \odot x_1^3 \oplus \tilde{1} \odot 3x_1^2h \oplus \tilde{1} \odot 3x_1h^2 \oplus \tilde{1} \odot h^3 \oplus \tilde{2} \odot x_2^3 \oplus \tilde{2} \odot 3x_2^2h \oplus$$
$$\tilde{2} \odot 3x_2h^2 \oplus \tilde{2} \odot h^3 \oplus \tilde{1} \odot x_1.x_2 \oplus \tilde{1} \odot x_1.h \oplus \tilde{1} \odot x_2.h \oplus \tilde{1} \odot h^2, \quad (3.35)$$

$$\tilde{f}(x_1 + h, x_2 + h) \oplus \tilde{f}(x_1, x_2) = \tilde{1} \odot 2x_1^3 \oplus \tilde{1} \odot 3x_1^2h \oplus \tilde{1} \odot 3x_1h^2 \oplus \tilde{1} \odot h^3 \oplus \tilde{2} \odot 2x_2^3 \oplus \tilde{2} \odot 3x_2^2h$$
$$\oplus \tilde{2} \odot 3x_2h^2 \oplus \tilde{2} \odot h^3 \oplus \tilde{1} \odot 2x_1x_2 \oplus \tilde{1} \odot h^2 \oplus \tilde{1} \odot x_2h \oplus \tilde{1} \odot x_1h, \quad (3.36)$$

$$\tilde{f}(x_1 + h, x_2) \oplus f(x_1, x_2 + h) = \tilde{1} \odot 2x_1^3 \oplus \tilde{1} \odot 3x_1^2h \oplus \tilde{1} \odot 3x_1h^2 \oplus \tilde{1} \odot h^3 \oplus \tilde{2} \odot 2x_2^3 \oplus \tilde{1} \odot 2x_1x_2$$
$$\oplus \tilde{1} \odot x_2h \oplus \tilde{2} \odot 3x_2^2h \oplus \tilde{2} \odot 3x_2h^2 \oplus \tilde{2} \odot h^3 \oplus \tilde{1} \odot x_1h. \quad (3.37)$$

From (3.35), (3.36) and (3.37), we deduce:

$$\tilde{a}_{12}(x_1, x_2) = \tilde{1}$$

. Subsequently, we obtain:

$$a_{12,\alpha}(x_1, x_2) = \int_0^1 (-4\alpha + 4)\ d\alpha.$$

Hence, $a_{12}(x_1, x_2) = 2$.
For this function, we have: $a_{12}(x_1, x_2) = a_{21}(x_1, x_2) = 2$.

Let's calculate the last component $a_{22}(x_1, x_2)$:

$$\tilde{f}(x_1, x_2 + 2h) = \tilde{1} \odot x_1^3 \oplus \tilde{2} \odot x_2^3 \oplus \tilde{2} \odot 6x_2^2h \oplus \tilde{2} \odot 12x_2h^2 \oplus \tilde{2} \odot 8h^3 \oplus \tilde{1} \odot x_1x_2 \oplus \tilde{1} \odot 2x_1h, \quad (3.38)$$

$$\tilde{f}(x_1, x_2+2h) \oplus \tilde{f}(x_1, x_2) = \tilde{1} \odot 2x_1^3 \oplus \tilde{2} \odot 2x_2^3 \oplus \tilde{2} \odot 6x_2^2h \oplus \tilde{2} \odot 12x_2h^2 \oplus \tilde{2} \odot 8h^3 \oplus \tilde{1} \odot 2x_1x_2 \oplus \tilde{1} \odot 2x_1h, \quad (3.39)$$

$$2 \odot \tilde{f}(x_1, x_2 + h) = \tilde{1} \odot 2x_1^3 \oplus \tilde{2} \odot 2x_2^3 \oplus \tilde{2} \odot 6x_2^2 h \oplus \tilde{2} \odot 6x_2 h^2 \oplus \tilde{2} \odot 2h^3 \oplus \tilde{1} \odot 2x_1 x_2 \oplus \tilde{1} \odot 2x_1 h. \quad (3.40)$$

From (3.38), (3.39) and (3.40), we deduce:

$$\tilde{a}_{22}(x_1, x_2) = \tilde{2} \odot 6x_2 \oplus \tilde{2} \odot 6h.$$

For h small enough, we have: $\tilde{a}_{22}(x_1, x_2) = \tilde{2} \odot 6x_2$.
Therefore we obtain:

$$a_{22,\alpha}(x_1, x_2) = [\alpha + 1, -\alpha + 3].6x_2.$$

So, we have:
$a_{22,\alpha}(x_1, x_2) = 6 \int_0^1 (-2\alpha + 2).x_2 \, d\alpha.$
Therefore, $a_{22}(x_1, x_2) = 6x_2$.
From which $A(x_1, x_2) = \begin{pmatrix} 12x_1 & 2 \\ & \\ 2 & 6x_2 \end{pmatrix}.$

By calculating the sequence of non-dominated solutions $\overline{x}^s$, $s = 0, 1, 2, 3, \cdots$, using (3.20), we obtain the data in Table 1.
We obtained the non-dominated solution $x^* = (0, 0)^T$ with a precision of 10^{-4} for the considered problem.

TABLE 1. Convergence of Newton's method with fuzzy gH-finite difference

s	x_1^s	x_2^s	$\tilde{f}(x_1^s, x_2^s)$	$\Upsilon(\tilde{f}(x_1^s, x_2^s))$
0	1	1	(-1,4,9)	4
1	0.44118	0.35294	(-0.17142, 0.32951, 0.85663)	0,33605
2	0.23930	-0.04954	(-0.00197, 0.00161, 0.00518)	0,00161
3	0.02407	0.13723	(-0.00073, 0.00849, 0.01771)	0,00849
4	0.02927	-0.00249	(0.00005, -0.00005, -0.00014)	-0,00005
5	0.00003	0.00257	(0.00000, 0.00000, -0.00000)	0
6	0.00000	0.00000	(-0.00000, 0.00000, 0.00000)	0

Based on the values of the ranking function and the reduced number of iterations in Table1, we conclude that for this problem, the proposed method yields better results than those presented in [23, 24].

Example 3.8. Consider the following fuzzy problem:

$$\min \tilde{f}(x_1, x_2) = \tilde{1} \odot x_1^2 \oplus \tilde{2} \odot x_1.x_2 \oplus \tilde{3} \odot x_2^2. \quad (3.41)$$

where $x_1, x_2 \in \mathbb{R}$, $\tilde{1} = (-1, 1, 3), \tilde{2} = (0, 1, 2)$ *and* $\tilde{3} = (1, 2, 4)$ are triangular fuzzy numbers.
Table 2 presents with a precision of 10^{-4}, the sequence of non-dominated solutions $\overline{x}^s$, $s = 0, 1, 2, 3, \cdots$, calculated using (3.20).

The optimal solution, $x^* = (0, 0)^T$, with an accuracy of 10^{-4}, is obtained at the first iteration because the objective function is quadratic. In this case, the proposed method yields the same results as those in ([23], [24]).

TABLE 2. Convergence of the example problem(3.8)

s	x_1^s	x_2^s	$\tilde{f}(x_1^s, x_2^s)$	$\Upsilon(\tilde{f}(x_1^s, x_2^s))$
0	2	-2	(0,8,20)	9
1	0	0	(0, 0, 0)	0

Example 3.9. Consider the following fuzzy problem:

$$\min \tilde{f}(x_1, x_2) = \tilde{2} \odot x_1^3 \oplus \tilde{2} \odot x_2^3 \oplus \tilde{3}. \tag{3.42}$$

Where $x_1, x_2 \in \mathbb{R}$, $\tilde{2} = (1, 2, 4)$ and $\tilde{3} = (1, 3, 5)$ are triangular fuzzy numbers.
Table 3 records the sequence of non-dominated solutions, x^s, $s = 0, 1, 2, 3, \cdots$, with a precision of 10^{-4}, calculated using (3.20). The minimizer of the objective function is $x^* = (0.00097, 0.00097)^T$ with a precision of 10^{-4}. The proposed method provides the same result for this function with the same number of iterations as the methods proposed in [23, 24] Newton.

TABLE 3. Convergence of the problem in Example 3.9

s	x_1^s	x_2^s	$\tilde{f}(x_1^s, x_2^s)$	$\Upsilon(\tilde{f}(x_1^s, x_2^s))$
0	1	1	(3,7,13)	7,5
1	0.50000	0.50000	(1.25000, 3.50000, 6.00000)	3,56250
2	0.25000	0.25000	(1.03125, 3.06250, 5.12500)	3,07031
3	0.12500	0.12500	(1.00391,3.00781,5.01562)	3,00879
4	0.06250	0.06250	(1.00049,0.00098, 5.00195)	1,5011
5	0.03125	0.03125	(1.00006,3.00012,5.00024)	3,00013
6	0.01562	0.01562	(1.00001, 3.00001, 5.00003)	3,00001
7	0.00781	0.00781	(1.00000, 3.00000, 5.00000)	3,00000
8	0.00390	0.00390	(1.00000, 3.00000, 5.00000)	3,00000
9	0.00195	0.00195	(1.00000, 3.00000, 5.00000)	3,00000
10	0.00097	0.00097	(1.00000, 3.00000, 5.00000)	3,00000

Example 3.10. Consider the following fuzzy problem:

$$\min \tilde{f}(x_1, x_2) = \widetilde{20} \odot x_1 \oplus \widetilde{26} \odot x_2 \oplus \widetilde{4} \odot x_1.x_2$$
$$\oplus (\widetilde{-4}) \odot x_1^2 \oplus (\widetilde{-3}) \odot x_2^2, \tag{3.43}$$

where $x_1, x_2 \in \mathbb{R}$, $\widetilde{20} = (19, 20, 21)$, $\widetilde{26} = (25, 26, 27)$, $\widetilde{4} = (2, 4, 6)$, $\widetilde{-4} = (-6, -4, -2)$ and $\widetilde{-3} = (-5, -3, -2)$ are triangular fuzzy numbers.
The sequence of non-dominated solutions x^s, where $s = 0, 1, 2, 3, \cdots$ is calculated using (4). The resulting Table4 is shown below: The minimizer of the problem is $x^* = (-0.12500, -0.25000)^T$ with an accuracy of

TABLE 4. Convergence of the problem in Example 3.10

s	x_1^s	x_2^s	$\tilde{f}(x_1^s, x_2^s)$	$\Upsilon(\tilde{f}(x_1^s, x_2^s))$
0	1	1	(35,43,50)	42,75
1	-0.12500	-0.25000	(-8.96875, -9.12500, -9.34375)	-9.14062

10^{-4}. This solution is obtained in the first iteration because the objective function is quadratic. Considering the value of the ranking function, we conclude that the proposed method is also better than those in [23,24] for this function.

3.5. **Discussion.** Considering the number of iterations required to obtain an optimal solution, as well as the values of the ranking function and the minimum objective function values, the proposed method yields better results. These observations demonstrate the effectiveness of the proposed method in solving nonlinear fuzzy optimization problems.

4. Conclusion

In this paper, an extension of Newton's method to fuzzy gH-finite difference for fuzzy nonlinear functions, twice gH-continuously differentiable or nondifferentiable, has been proposed. This method uses fuzzy gH-finite differences to approximate the gradient and Hessian matrix of any fuzzy nonlinear function. It is powerful and easy to use because there is no need to calculate the gradient and Hessian matrix using traditional methods. Four multivariate nonlinear fuzzy optimization problems were also solved. The satisfactory results show that the proposed method is more suitable for unconstrained multivariate fuzzy nonlinear optimization problems. Thus, the results of this study improve upon the fuzzy Newton methods presented in previous literature.

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